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NUMERICAL EXPERIMENTS CONCERNING THE EIGENVALUES OF THE LAPLACIAN ON A ZOLL SURFACE

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0. Introduction

A Zoll surface is a surface of revolution in $\mathfrak{R}^3$ diffeomorphic to S, the unit 2-sphere, with the property that all geodesics are closed with period 2π. S is, for example, a Zoll surface. O. Zoll [7] first proved the existence of a Zoll surface not isometric to S.

Eigenvalues of Δ, the Laplacian defined on functions whose domain is S, are $l(l+1)$ with multiplicities $2l+1$ ($l = 0, 1, 2, \cdots$). For Zoll surfaces other than S, each multiple eigenvalue $l(l+1)$ splits into a "cluster" of eigenvalues near $l(l+1)$. Specifically, a result of Alan Weinstein [5] states that there is a number $M > 0$ independent of l such that the eigenvalues of the l-th cluster are contained in the interval $[l(l+1) - M, l(l+1) + M]$.

In this work the following question is addressed: for a Zoll surface what is the structure of the l-th cluster for large l? Numerical experiments were made where by the eigenvalues of the first 15 or 20 clusters of selected Zoll surfaces were approximated. These computations led to two conjectures. Conjecture 1: the arithmetic mean of the eigenvalues in the l-th cluster approaches $l(l+1)$ as l goes to ∞. Conjecture 2: the distribution of eigenvalues in the l-th cluster approaches (in a sense to be made clear) a limiting function as l goes to ∞.

In an attempt to explain these experimental results Weinstein [6] proved a theorem. The theorem which concerns the cluster structure of Δ plus a potential function tends to corroborate the conjecture.

In §1 a method for constructing one parameter families of Riemannian manifolds isometric to Zoll surfaces is given. In §2 the Laplace-Beltrami operators corresponding to the Riemannian manifolds of §1 are written in geographic coordinates, and by separation of variables an ordinary differential operator D_ε^m is defined. In §3 cluster is defined, and information about eigenvalues of D_0^m and Δ is given. In §4 the method by which the eigenvalues

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are approximated is given. Conjectures 1 and 2 are stated exactly in §5. In §6 the numerical experiments are described. §7 contains information about the accuracy of the approximations. In §§8 and 9 evidence for conjectures 1 and 2 are given, respectively. In §10 results for non-Zoll experiments is presented. In the appendix EIGEN 2, the subroutine that computed the elements of the matrices which approximate the D_ε^m's is described and listed.

The eigenvalues computed in the experiments do not appear in this work, but the author will send a list of these eigenvalues upon request.

We thank Alan Weinstein for suggesting these experiments.

1. Construction of one-parameter families of Zoll metrics

A Riemannian metric g is a Zoll metric if the Riemannian manifold (S, g) is isometric to a Zoll surface. We will now give a method for constructing one-parameter families of Zoll metrics g_ε such that g_0 is the canonical metric on S.

Consider the map $X : [0, 2\pi] \times [0, \pi] \to \mathfrak{R}^3$ given by

$$(v, u) \to (\sin u \cos v, \sin u \sin v, \cos u).$$

S is the image of X. Let $U \subset S$ be the image of $(0, 2\pi) \times (0, \pi)$ under X. U is open and dense in S. Define functions ϕ and θ on U as follows: $X(\phi(m), \theta(m)) = m$ for $m \in U$. (ϕ, θ) is the geographic coordinate system on U; $\phi(m)$ is the longitude of m, and $\theta(m)$ is the latitude of m measured from the "north pole".

A smooth function $\sigma : [-1, 1] \to \mathfrak{R}$ which is zero at the end points will be called a perturbing function. For each ε in some neighborhood of zero let h_ε be a positive valued function on $[0, \pi]$ given by

$$h_\varepsilon(\theta) = 1 + \varepsilon\sigma(\cos\theta).$$

Given a perturbing function σ and allowing ε to range over an appropriate neighborhood of zero we define a one-parameter family g_ε of metrics on U:

$$(*) \qquad g_\varepsilon = h_\varepsilon^2(\theta)d\theta^2 + \sin^2\theta d\phi^2.$$

A. L. Besse [1] states that g_ε can be extended to a metric on S if the function h_ε has the following properties: h_ε is smooth on $[0, \pi]$, and $h_\varepsilon(0) = h_\varepsilon(\pi) = 1$. h_ε has those properties. Also Besse shows that if the perturbing function σ is odd, that is, $\sigma(-x) = -\sigma(x)$, then g_ε is a Zoll metric.

Thus g_ε as defined by $(*)$ is a family of metrics on S for arbitrary σ, and is a family of Zoll metrics if σ is an odd function.

2. Zoll perturbations of the Laplace-Beltrami operator

For each Riemannian manifold there is a unique Laplace-Beltrami operator. We will call this operator the Laplacian, and its domain is the set of smooth function on the manifold. The Laplacian on $\mathcal{R}^2$ with the canonical metric is

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

We are interested in certain perturbations of Δ, the Laplacian on (S, g_0). Given σ, a perturbing function, define Δ_ε, a perturbation of Δ, to be the Laplacian on (S, g_ε). g_ε is defined by (*) of §1 with perturbation function σ. If σ is odd, then we call Δ_ε a Zoll perturbation of Δ. Thus a Zoll perturbation of Δ is generated by a Zoll metric.

If F is a smooth function on S, then Δ_ε in geographic coordinates is given by

$$\Delta_\varepsilon[F]=\frac{-1}{h_\varepsilon^2(\theta)}\frac{\partial^2 F}{\partial\theta^2}+\left(\frac{h_\varepsilon'(\theta)}{h_\varepsilon^3(\theta)}-\frac{\cot\theta}{h_\varepsilon^2(\theta)}\right)\frac{\partial F}{\partial\theta}+\frac{-1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

This result can be seen by using the definition of Δ_ε given by Warner [4].

For each integer m we will define D_ε^m, a second order ordinary differential operator. The domain of D_ε^m is the set of smooth functions f on $[0, \pi]$:

$$D_\varepsilon^m[f](\theta)=\frac{-1}{h_\varepsilon^2(\theta)}f''(\theta)+\left(\frac{h_\varepsilon'(\theta)}{h_\theta^3(\theta)}-\frac{\cot\theta}{h_\varepsilon^2(\theta)}\right)f'(\theta)+\frac{m^2}{\sin^2\theta}f(\theta).$$

Theorem. *λ is an eigenvalue of Δ_ε if and only if there is an m such that λ is an eigenvalue of D_ε^m.*

A proof can be constructed by using two facts. One fact is that if F is smooth on S, then F is the (possibly infinite) sum of terms of the form $u_m(\phi)f(\theta)$. $f(\theta)$ is smooth on $[0, \pi]$, and

$$u_m(\phi)=\begin{cases}\sin(m\phi) & \text{if } m<0,\\ 1 & \text{if } m=0,\\ \cos(m\phi) & \text{if } m>0.\end{cases}$$

The other fact is that

$$\Delta_\varepsilon[F]=u_m(\phi)D_\varepsilon^m[f][\theta],$$

if $F=u_m(\phi)f(\theta)$.

By means of the above theorem we will investigate the eigenvalues of Δ_ε by examining the eigenvalues of the D_ε^m.

There are two notable facts about the boundary conditions of the eigenfunctions of D_ε^m: (1) if $m \neq 0$, the eigenfunctions are zero at the end points, and (2) the derivatives of the eigenfunctions of D_ε^0 are zero at the end points. To prove the first fact we assume that f is an eigenfunction of $D_\varepsilon^m (m \neq 0)$, and obtain the following:

$$\lambda f(\theta) \sin^2 \theta = D_\varepsilon^m[f](\theta) \sin^2 \theta.$$

Expanding the right side of the above equation we obtain

$$\left(\frac{-\sin^2 \theta}{h_\varepsilon^2(\theta)}\right) f''(\theta) + \left(\frac{h_\varepsilon'(\theta) \sin^2 \theta}{h_\varepsilon^3(\theta)} - \frac{\cos \theta \sin \theta}{h_\varepsilon^2 \theta}\right) f'(\theta) + m^2 f(\theta).$$

Therefore $f(\theta) = 0$ when θ equals 0 or π.

To prove the second fact we assume that f is an eigenfunction of D_ε^0, and obtain an expression similar to the above:

$$\lambda f(\theta) \sin \theta = D_\varepsilon^0[f](\theta) \sin \theta.$$

We expand the right side of the above equation

$$\left(\frac{-\sin \theta}{h_\varepsilon^2(\theta)}\right) f''(\theta) + \left(\frac{h_\varepsilon'(\theta) \sin \theta}{h_\varepsilon^3(\theta)} - \frac{\cos \theta}{h_\varepsilon^2(\theta)}\right) f'(\theta).$$

Therefore $f'(\theta) = 0$ when θ equals 0 or π.

It should also be noted that $D_\varepsilon^m = D_\varepsilon^{-m}$.

3. Cluster of eigenvalues

Henceforth m is an integer, and k and l are nonnegative integers. We make the following definitions:

$$\mu(l) = l(l+1),$$
$$\lambda_\varepsilon(k) = \text{the } k\text{-th eigenvalue of } \Delta_\varepsilon,$$
$$\lambda_\varepsilon(k, m) = \text{the } k\text{-th eigenvalue of } D_\varepsilon^m.$$

Vilenkin [3] treats the eigenvalues and eigenfunctions of D_0^m. Letting $x = \cos \theta$, we can write D_0^m as

$$(x^2 - 1)\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{m^2}{1 - x^2}.$$

The associated Legendre function $P_l^m(x)(|m| \leqslant l)$ is the eigenfunction of D_0^m with value $\mu(l)$, and $\mu(l)$ is a simple eigenvalue. Thus $\lambda_0(k, m) = \mu(k + |m|)$.

The eigenvalues of Δ_0 are $\mu(l)$, and have multiplicity $2l + 1$. The eigenfunctions of Δ_0 belonging to $\mu(l)$ are $u_m(\phi)P_l^m(\cos \theta)$, where $|m| \leqslant l$.

Below are listed the first few members of the spectrum of the specified operators.

Δ_0:	0,	2,	2,	2,	6,	6,	6,	6,	6,	12,	12, $\cdots$
D_0^0:	0,	2,	6,	12,	20,	30,	42,	56,	72,	$\cdots$	
D_0^3:	12,	20,	30,	42,	56,	72,	90,	110,	$\cdots$		
D_0^{-2}:	6,	12,	20,	30,	42,	56,	72,	90,	$\cdots$.		

From the above we see that if $l^2 \leqslant k < (l+1)^2$, then $\lambda_0(k) = \mu(l)$. Define the *l-th order cluster* of Δ_ε to be $\{\lambda_\varepsilon(k) : l^2 \leqslant k < (l+1)^2\}$. $\mu(l)$ is the only member of the l-th order cluster of Δ_0.

Define the *l-th cluster* of Δ_ε to be $\{\lambda_\varepsilon(k, m) : l = k + |m|\}$. $\mu(l)$ is the only member of the l-th cluster of Δ. As Δ_ε is perturbed from $\varepsilon = 0$, $\mu(l)$ splits into the eigenvalues of the l-th cluster.

Both the l-th cluster and the l-th order cluster are subsets of spectrum of Δ_ε, and are equal when $\varepsilon = 0$. If $\varepsilon \neq 0$, the two subsets may not be equal; the clusters may "over lap", where as order clusters never "over lap".

A result of A. Weinstein [5] states the following: if Δ_ε is a Zoll perturbation, then there is an $M > 0$ such that for all l the l-th order cluster is contained in the interval $[\mu(l) - M, \mu(l) + M]$, where M depends only on g_ε. Using this result one can show that eventually the order clusters are equal to the clusters when Δ_ε is a Zoll perturbation.

The purpose of this paper is to study the structure of the eigenvalues in the l-th cluster of Δ_ε, a Zoll perturbation, as l approaches ∞.

4. The method of approximating eigenvalues

In this section we give the method by which we numerically approximate the eigenvalues of Δ_ε. To achieve our goal we actually compute approximations to the eigenvalues of D_ε^m.

The first step in our method is to define E_n for D_ε^m, where n is a positive integer. E_n is an $(n-1) \times (n-1)$ matrix if $m \neq 0$, and E_n is an $(n+1) \times (n+1)$ matrix if $m = 0$. Define $\Lambda_\varepsilon^n(k, m)$ to be the k-th eigenvalue of $E_n (k < n)$. E_n will have the following important property:

$$\lim_{n \to \infty} \Lambda_\varepsilon^n(k, m) = \lambda_\varepsilon(k, m).$$

We will approximate $\lambda_\varepsilon(k, m)$ by $\Lambda_\varepsilon^n(k, m)$ using large values of n. We will, in general, calculate the first 15 or 20 eigenvalues of E_n, for $n = 1000$ or $n = 2000$.

The method of finite centered difference is used to define E_n. We divide $[0, \pi]$ into n subintervals of equal length; θ_i's are the end points of the subintervals. The $(n + 1)$ vector $(f_0, f_1, f_2, \ldots, f_n)$ approximates f, an eigenfunction of D_ε^m, by having $f_i \sim f(\theta_i)$. $f'(\theta_i)$ and $f''(\theta_i)$ are approximated by

$$f'(\theta_i) \sim \frac{f_{i+1} - f_{i-1}}{2\delta}, \quad f''(\theta_i) \sim \frac{f_{i+1} - 2f_i + f_{i-1}}{\delta^2},$$

where δ is the mesh size, the length of the subintervals. We now proceed formally to define E_n. Define the following expressions:

$$\begin{aligned} A(\theta) &= m^2/\sin^2\theta, \\ B(\theta) &= h_\varepsilon'(\theta)/h_\varepsilon^3(\theta) - \cot\theta/h_\varepsilon^2(\theta), \\ C(\theta) &= -1/h_\varepsilon^2(\theta), \\ \theta_i &= i\pi/n \quad (i = 0, 1, \cdots, n), \\ \delta &= \pi/n = \text{the mesh size}, \\ A_i &= A(\theta_i), \quad B_i = B(\theta_i), \quad C_i = C(\theta_i). \end{aligned}$$

$A(\theta)$, $B(\theta)$, and $C(\theta)$ are the coefficients of $f(\theta)$, $f'(\theta)$, and $f''(\theta)$, respectively, in the definition of $D_\varepsilon^m[f](\theta)$.

Set up difference equations for $i = 1, \cdots, n - 1$:

$$A_i f_i + B_i \frac{f_{i+1} - f_{i-1}}{2\delta} + C_i \frac{f_{i+1} - 2f_i + f_{i-1}}{\delta^2} = \lambda f_i.$$

The left side of the equation approximates $D_\varepsilon^m[f](\theta_i)$. The right approximates $\lambda f(\theta_i)$. By combining like terms the equation reduces to

$$(**) \qquad \alpha_i f_{i-1} + \beta_i f_i + \gamma_i f_{i+1} = \lambda f_i,$$

where

$$\alpha_i = \frac{C_i}{\delta^2} - \frac{B_i}{2\delta}, \quad \beta_i = A_i - 2\frac{C_i}{\delta^2}, \quad \gamma_i = \frac{C_i}{\delta^2} + \frac{B_i}{(2\delta)}.$$

Call (**) the i-th difference equation.

At this point we must consider two cases for D_ε^m: $m \neq 0$ and $m = 0$. We proceed with case $m \neq 0$. As mentioned, $f(0) = f(\pi) = 0$, when f is a eigenfunction of D_ε^m and $m \neq 0$. So define $f_0 = f_n = 0$.

Because $f_0 = 0$ the first difference equation is

$$\beta_1 f_1 + \gamma_1 f_2 = \lambda f_1,$$

and because $f_n = 0$ the $(n - 1)$-st difference equation is

$$\alpha_{n-1} f_{n-2} + \beta_{n-1} = \lambda f_{n-1}.$$

Using matrix notation we write the $(n-1)$ difference equations as follows:

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & & 0 & \\ 0 & \alpha_3 & \beta_3 & & & & \\ & & & \ddots & & & \\ & & & & \beta_{n-3} & \gamma_{n-3} & 0 \\ & 0 & & & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\ & & & & 0 & \alpha_{n-1} & \beta_{n-1} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \end{bmatrix}.$$

The tridiagonal matrix is the above mentioned E_n. Thus E_n is an $(n-1) \times (n-1)$ tridiagonal matrix with $\beta_1, \cdots, \beta_{n-1}$ on the main diagonal, $\alpha_2, \cdots, \alpha_{n-1}$ on the subdiagonal, $\gamma_1, \cdots, \gamma_{n-2}$ on the super diagonal, and zero everywhere else. By standard approximation theory $\lim_{n\to\infty} \Lambda_\varepsilon^n(k, m) = \lambda_\varepsilon(k, m)$. (See Isaacson and Keller [2].)

Next consider the case $m = 0$. $f'(0) = f'(\pi) = 0$ when f is an eigenfunction of D_ε^0. Notice that $m = 0$ implies $A(\theta) = 0$. We must still approximate f_0 and f_n.

Let us calculate $D_\varepsilon^0[f](\pi)$ where f is an eigenfunction. Keep in mind that $B(\theta)$ has singularities at $\theta = 0$ and $\theta = \pi$.

$$\begin{aligned} D_\varepsilon^0[f](\pi) &= \lim_{\theta\to\pi} (B(\theta)f'(\theta)) + C(\pi)f''(\pi) \\ &= \lim_{\theta\to\pi} \left(\frac{f'(\theta)}{\sin\theta}\right)\cos\pi + (-1)f''(\pi). \end{aligned}$$

The second line is true by the definitions of $B(\theta)$ and $C(\theta)$, and because $h_\varepsilon(\pi) = 1$ and $h_\varepsilon'(\pi) = 0$. L'Hosptal's rule gives

$$D_\varepsilon^0[f](\pi) = -\frac{f''(\pi)}{\cos\pi} - f''(\pi) = -2f''(\pi).$$

Similar calculations yield $D_\varepsilon^0[f](0) = -2f''(0)$. We approximate $f''(\pi)$ as follows:

$$f''(\pi) \sim \frac{f'(\pi) - f'(\pi - \delta/2)}{\delta/2}.$$

Because $f'(\pi) = 0$ and

$$f'(\pi - \delta/2) \sim \frac{f_n - f_{n-1}}{\delta},$$

we make the following approximation

$$f'(\pi) \sim \frac{0 - \dfrac{f_n - f_{n-1}}{\delta}}{\delta/2} = -2\frac{f_n - f_{n-1}}{\delta^2}.$$

Combining the above we have

$$D_\varepsilon^0[f](\pi) = -2f''(\pi) \sim \frac{4}{\delta^2}f_n - \frac{4}{\delta^2}f_{n-1}.$$

Define the n-th difference equation to be

$$-\frac{4}{\delta^2}f_{n-1} + \frac{4}{\delta^2}f_n = \lambda f_n.$$

Similarly the 0-th difference equation is defined to be

$$\frac{4}{\delta^2}f_0 - \frac{4}{\delta^2}f_1 = \lambda f_0.$$

If we let $q = 4/\delta^2$, $\beta_0 = q$, $\gamma_0 = -q$, $\alpha_n = -q$, and $\beta_n = q$, then E_n is the $(n+1) \times (n+1)$ tridiagonal matrix

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & & & & \\ \alpha_1 & \beta_1 & \gamma_1 & & 0 & & \\ 0 & \alpha_2 & \beta_2 & & & & \\ & & & \ddots & & & \\ & & & & \beta_{n-2} & \gamma_{n-2} & 0 \\ & 0 & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\ & & & & 0 & \alpha_n & \beta_n \end{bmatrix}.$$

Again $\lim_{n\to\infty} \Lambda_\varepsilon^n(k, 0)$-$\lambda_\varepsilon(k, 0)$.

The second step of our method is to calculate the eigenvalues of E_n. This step is achieved by the use of EQRTIS. EQRTIS is a FORTRAN subroutine that computes the j smallest eigenvalues of a tridiagonal matrix. An important feature of EQRTIS is that it computes the eigenvalues to the precision of the computer, which is about 15 decimal places.

EQRT1S is a product of Internatiolnal Mathematical and Statistical Libraries, Inc. (IMSL) of Houston, Texas, and is a part of the University of California, Berkeley computer library.

5. The statements of Conjecture 1 and Conjecture 2

Recall that the l-th cluster of Δ_ε is defined to be $\{\lambda_\varepsilon(k, m) : k + |m| = l\}$, and that, as an eigenvalue of D_ε^m, $\lambda_\varepsilon(k, m)$ is simple.

The l-th cluster of Δ contains only $\mu(l)$. The l-th cluster of Δ_ε has $2l + 1$ members, counting multiplicities.

Define $\bar{\lambda}_\varepsilon(l)$ to be the mean of the l-th cluster of Δ_ε:

$$\bar{\lambda}_\varepsilon(l) = \frac{\sum_{m=-l}^{l} \lambda(l - |m|, m)}{2l + 1}.$$

Conjecture 1. *If Δ_ε is a Zoll perturbation of Δ, then $|\mu(l) - \bar{\lambda}_\varepsilon(l)|$ approaches zero as l goes to infinity.*

For the l-th cluster of Δ_ε we define a real valued function G_l whose domain is the interval $[-1, 1]$. We make the following definitions:

$$x_m = m/l \quad (|m| \leqslant l),$$
$$y_m = \lambda_\varepsilon(l - |m|, m) - \mu(l),$$
$$G_l(x_m) = y_m.$$

If $x_m \leqslant x \leqslant x_{m+1}$, then

$$G_l(x) = y_m + \frac{y_{m+1} - y_m}{x_{m+1} - x_m}(x - x_m).$$

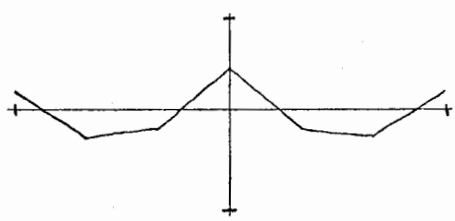
Thus the value of $G_l(x)$ when $x_m \leqslant x \leqslant x_{m+1}$ is found by linear interpolation. This definition is not valid for $G_0(x)$; for completeness let $G_0(x) = 0$ for all x in $[-1, 1]$. (For every perturbation of Δ the 0-th cluster is $\{\mu(0)\}$.) Because $\lambda_\varepsilon(k, m) = \lambda_\varepsilon(k, -m)$, G_l is always an even function.

Consider G_3 for a certain perturbation of Δ. We use Table 5. (Keep in mind that $\mu(3) = 13$.)

m	$\lambda_\varepsilon(3 - \lvert m\rvert, m)$	x_m	y_m
0	12.55	0	0.55
± 1	11.77	$\pm 1/3$	−0.23
± 2	11.65	$\pm 2/3$	−0.35
± 3	12.22	$+1$	0.22

TABLE 5

The graph of G_3 follows.



Conjecture 2. *If Δ_ε is a Zoll perturbation of Δ, then there is a function, call it G_∞, on $[-1, 1]$ such that $\|G_l - G_\infty\|$ approaches zero as l goes to infinity, where $\| \|$ is the sup-norm.*

The result of Weinstein mentioned in §3 implies that G_∞ is bounded.

Clearly, both conjectures are true for Δ_ε when $\varepsilon = 0$.

6. Nature of the experiments

An experiment is a set of computer calculations which approximate eigenvalues of some perturbation of Δ.

In the FORTRAN program used to do the calculations the input parameters are FACTOR, IFIRST, M, N, and SIG(X). FACTOR is a number in the interval [0, 1). IFIRST is a positive integer. M is a nonnegative integer corresponding to $|m|$, where m is the parameter for the operator D_ε^m. N is a positive integer corresponding to n, where π/n is the mesh size. SIG(X) corresponds to $\sigma(x)$, the perturbing function.

g_ε, as defined by (*), is a Riemannian metric for S when ε is a member of a certain maximal open interval about zero. Let ε_0 be an endpoint of the interval nearest zero. This would imply that g_{ε_0} is not a Riemann metric, and that g_ε is when $|\varepsilon| < |\varepsilon_0|$. For a Zoll metric the endpoints are equi-distant from zero.'

There are two types of experiments, short and full. A short experiment uses all input parameters. The outputs are $\Lambda_\varepsilon^N(k, M)$ where $0 \leqslant k <$ IFIRST, where $\varepsilon = \varepsilon_0$(FACTOR), and where SIG($X$) is the perturbing function for Δ_ε. A short experiment therefore approximates the first IFIRST eigenvalues of D_ε^m.

In a full experiment FACTOR, IFIRST, N, and SIG(X) are inputs, but M is not. The outputs are $\Lambda_\varepsilon^n(k, m)$, where $M \geqslant 0$ and $M + k <$ IFIRST. FACTOR and SIG(X) have the same role as in a short experiment. A full experiment approximates the first IFIRST clusters of Δ_ε

Table 6 lists experiments, their types, and their inputs. Each experiment is named by a number. In an experiment with FACTOR = 0 the eigenvalues are for Δ are approximated, and therefore SIG(X) is not given.

N is not listed for experiment 11. The reason being that different values of N were used. The $\Lambda_\varepsilon^N(k, M)$ were calculated for $k + M <$ IFIRST = 40. The values of N are as follows: $N = 2000$ for $M = 0, \cdots, 30$; $N = 1000$ for $M = 31, \cdots, 34$; $N = 500$ for $M = 35, \cdots, 39$.

Zoll experiments are experiments in which SIG(X) is an odd function. Experiments in which FACTOR = 0 are automatically Zoll experiments.

Experiment Number	Type of Experiment	N	M	FACTOR	IFIRST	SIG(X)
1	full	2000		0.0	20	
2	full	2000		0.1	20	$X^3 - X$
3	full	2000		0.2	20	$X^3 - X$
4	full	2000		0.4	20	$X^3 - X$
5	full	2000		0.8	20	$X^3 - X$
6	full	1000		0.0	15	
7	full	1000		0.9999	15	$X^3 - X$
8	full	1000		0.9	9	$X^2 - 1$
9	full	2000		0.4	20	$(1 - X^2)$ $\cdot(X^3 - .4X)$
10	full	2000		0.4	20	$1 - X^2$
11	full			0.4	40	$X^3 - X$
12	short	2000	0	0.0	40	
13	full	1000		0.8	9	$X^3 - X$

TABLE 6. LIST OF EXPERIMENT AND INPUTS

7. Considerations on the accuracy of approximations

Two considerations contribute to our knowledge of the accuracy of our approximations to $\lambda_\varepsilon(k, m)$.

Basically, we are interested in the truncation error, $|\lambda_\varepsilon(k, m) - \Lambda_\varepsilon^n(k, |m|)|$. When can the truncation errors be known exactly? Only when the $\lambda_\varepsilon(k, m)$ are known, and that is only when $\varepsilon = 0$. Recall $\lambda_0(k, m) = \mu(k + |m|)$.

Consideration 1. *In all experiments with* FACTOR $= 0$ *the following is true*:

$$|\mu(l) - \Lambda_0^n(l, 0)| = \max\{|\mu(l) - \Lambda_0^n(k, |m|)|: k + |m| = l\}.$$

The consideration says that among the computed approximations to the members of the l-th cluster of Δ, $\Lambda_0^n(l, 0)$ has the greatest deviation from $\mu(l)$.

Consideration 2. *It will be assumed that* $|\lambda_\varepsilon(k, m) - \Lambda_\varepsilon^n(k, |m|)|$ *is near* $|\lambda_0(k, m) - \Lambda_0^n(k, |m|)|$.

Table 7 lists the errors $\mu(l) - \Lambda_0^n(l, 0)$ generated by experiments. In experiment 1, $n = 2000$ and $l = 0, \cdots, 19$, and in experiment 6, $n = 1000$ and $l = 0, \cdots, 14$. The table should give us an upper estimate of the truncation errors.

l	$\mu(l)$	Experiment 6 $\mu(l) - \Lambda_0^{1000}(l, 0)$	Experiment 1 $\mu(l) - \Lambda_0^{2000}(l, 0)$
0	0	.00000	.00000
1	2	.00000	.00000
2	6	.00003	.00001
3	12	.00011	.00003
4	20	.00032	.00008
5	30	.00072	.00018
6	42	.00142	.00036
7	56	.00254	.00063
8	72	.00421	.00105
9	90	.00660	.00165
10	110	.00987	.00247
11	132	.01423	.00356
12	156	.01990	.00497
13	186	.02710	.00678
14	210	.03611	00903
15	240		.00118
16	272		.01516
17	306		.01919
18	342		.02398
19	380		.02963

TABLE 7. $\mu(l) - \Lambda_0^n(l, 0)$ for Experiments 1 and 6.

8. Evidence for Conjecture 1

Evidence for the conjectures was generated by the Zoll experiments with FACTOR $\neq 0$. Define the computed mean,

$$\bar{\Lambda}_\varepsilon^n(l) = \frac{\Lambda_\varepsilon^n(l) + 2\Sigma_{m=1}^{l} \Lambda_\varepsilon^n(l - m, m)}{2l + 1}.$$

To support Conjecture 1 we will examine the computed mean deviation (CMD), $\mu(l) - \bar{\Lambda}_\varepsilon^n(l)$. (Note the absence of absolute value signs.)

Table 8 lists the CMD's of the relevant experiments. In columns one and two are the cluster numbers, l, and the $\mu(l)$ respectively. Above the remaining columns are experiment numbers, and below the experiment numbers are the CMD's for that experiment. So, for example, to find the CMD for the 7-th cluster of experiment 5, look in the column under 5 and on line $l = 7$.

For experiment 11 only the CMD's for $l = 20, \cdots, 39$ are listed. The reason is that experiments 11 and 4 are identical for $l = 0, \cdots, 19$. Therefore it would be repetitious to have CMD's listed again in the table.

Below are some observations about Table 8. The remarks after the observations are interpreted to be evidence for Conjecture 1.

Observation 1. In each experiment, $\mu(l) - \overline{\Lambda}(l)$ is eventually very small compared to $\mu(l)$.

Remark. If Conjecture 1 is true, then $\mu(l) - \overline{\Lambda}_\varepsilon^n(l)$ approaches $\overline{\lambda}_\varepsilon(l) - \overline{\Lambda}_\varepsilon^n(l)$. But $\overline{\lambda}_\varepsilon(l) - \overline{\Lambda}_\varepsilon^n(l)$ is small.

Observation 2. A CMD of experiment 5 is closer to zero than the corresponding CMD of experiment 13. For $l = 0, \cdots, 8$ these two experiments approximate the same eigenvalues, but for experiment 5, $n = 2000$ and for experiment 13, $n = 1000$.

l	μ	2	3	4	9	5	13	7	l	$\mu(l)$	11
0	0	.00000	.00000	.00000	.00000	.00000	.00000	.00000	20	420	.01699
1	2	.00085	.00333	.01223	.00766	.03558	.03558	.04450	21	462	.02056
2	6	.00002	.00020	.00315	.01791	.03928	.03931	.07039	22	506	.02466
3	12	.00001	.00005	.00084	.00460	.02097	.02106	.06109	23	552	.02935
4	20	.00003	.00003	.00039	.00190	.02143	.02165	.05008	24	600	.03466
5	30	.00007	.00008	.00020	.00140	.02136	.02184	.07565	25	650	.04068
6	42	.00013	.00014	.00022	.00071	.01221	.01310	.07733	26	702	.04743
7	56	.00029	.00026	.00032	.00073	.01611	.01765	.05934	27	756	.05501
8	72	.00040	.00042	.00052	.00060	.01155	.01407	.08790	28	812	.06364
9	90	.00063	.00066	.00078	.00088	.00985		.09069	29	870	.07284
10	110	.00095	.00099	.00118	.00119	.01132		.07774	30	930	.08323
11	132	.00136	.00142	.00169	.00171	.00878		.09977	31	992	.09471
12	156	.00190	.00199	.00235	.00237	.00964		.11201	32	1056	.10740
13	182	.00258	.00270	.00320	.00320	.01073		.10622	33	1122	.12148
14	210	.00360	.00361	.00426	.00425	.01039		.11919	34	1190	.13699
15	240	.00450	.00471	.00556	.00554	.01295			35	1260	.15430
16	272	.00570	.00606	.00714	.00709	.01455			36	1332	.17375
17	306	.00732	.00766	.00818	.00896	.01683			37	1406	.19586
18	342	.00914	.00957	.01127	.01118	.02064			38	1482	.22141
19	380	.01129	.01181	.01392	.01379	.02408			39	1560	.25034

TABLE 8. Computed Mean Deviations for Zoll Experiments.

Remark. The observation is consistent with Conjecture 1 for the following reason: $\bar{\Lambda}_\varepsilon^{2000}(l)$ is a better approximation to $\bar{\lambda}_\varepsilon(l)$ that $\bar{\Lambda}_\varepsilon^{1000}(l)$. So, if $|\mu(l) - \bar{\lambda}_\varepsilon(l)|$ is near zero, then $\bar{\Lambda}_\varepsilon^{2000}(l)$ should be nearer to $\mu(l)$ than $\bar{\Lambda}_\varepsilon^{1000}(l)$.

Observation 3. In each experiment, $\mu(l) - \bar{\Lambda}_\varepsilon^n(l)$ is eventually less than $\mu(l) - \Lambda_0^n(l, 0)$. (See Table 7.)

Remark. $\mu(l) - \bar{\Lambda}_\varepsilon^n(l)$ approaches $\bar{\lambda}_\varepsilon(l) - \bar{\Lambda}_\varepsilon^n(l)$ by Conjecture 1. So $\mu(l) - \bar{\Lambda}_\varepsilon^n(l)$ approaches $\bar{\lambda}_k 0(l) - \bar{\Lambda}_0^n(l)$ by Consideration 2. $\bar{\lambda}_0(l) = \mu(l)$. Therefore by Consideration 1, eventually $|\mu(l) - \bar{\Lambda}_\varepsilon^n(l)| < |\mu(l) - \Lambda_0^n(l, 0)|$.

9. Evidence for Conjecture 2

The evidence for Conjecture 2 is contained in the graphs which approximate some of the G_l's.

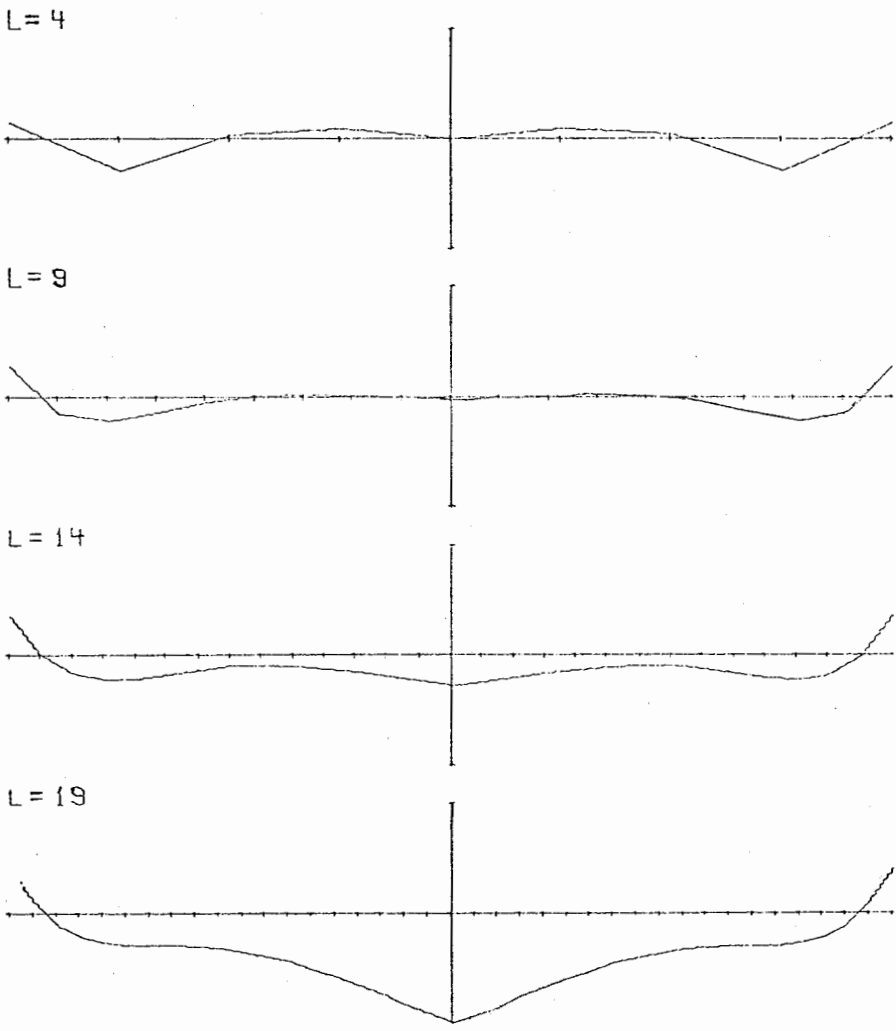
The graphs of this section were generated by output from Experiments 2, 3, 4, 5, 7, 9. Only the graphs from one experiment will be on any one page. At the top of each page will be the number of the experiment; to the left of each graph will be the cluster number. The y-axis is scaled so that the "top" and the "bottom" of the y-axis represents a certain distance from the origin. That distance is given by Table 9.

Experiment	2	3	4	5	7	9
Distance	.029	.048	.401	1.052	2.499	.353

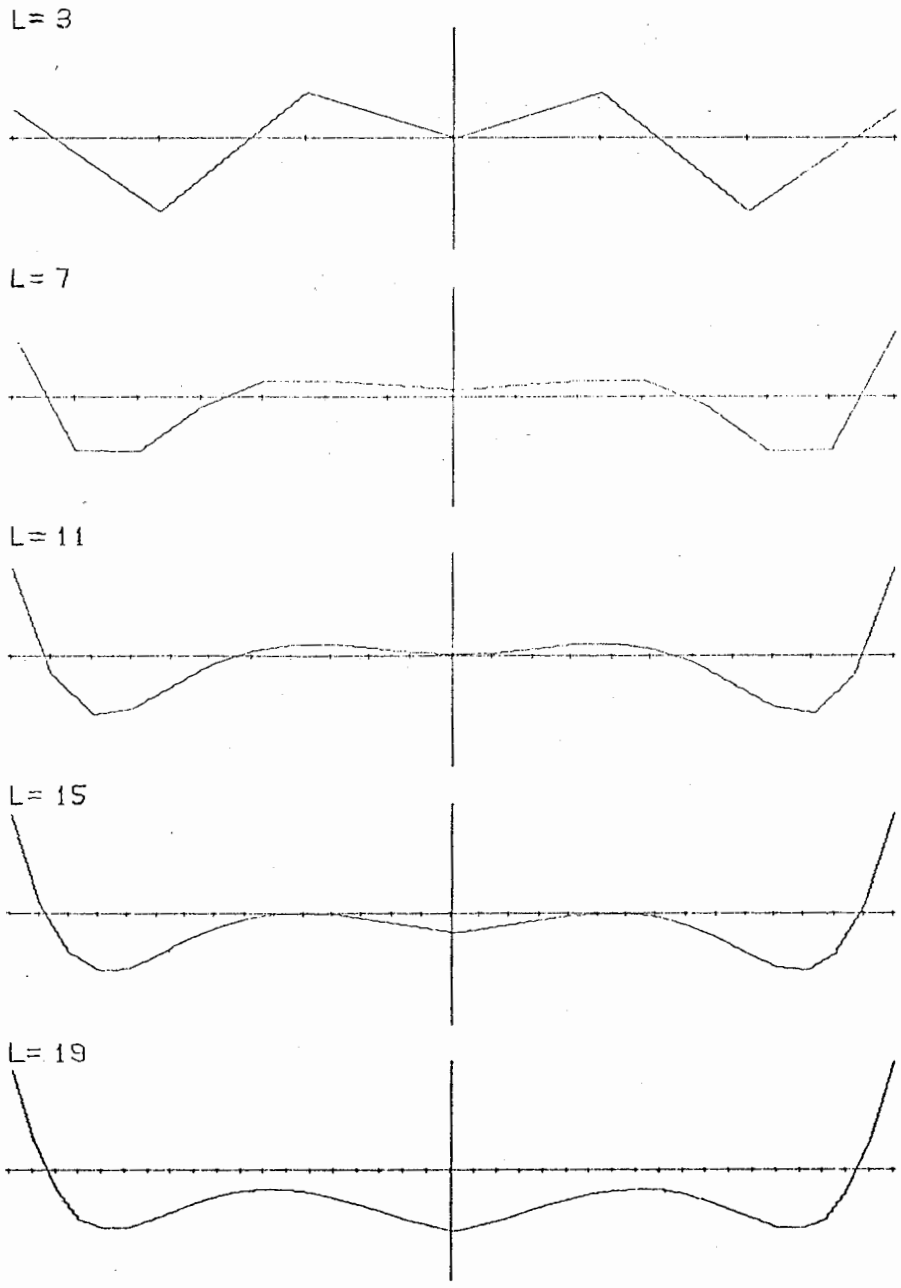
TABLE 9

Comments about experiments 2 and 3 should be made. The graphs of experiments 2 and 3 seem to "dip" in the center as l becomes larger. The phenomenon is caused by two factors: (1) as l increases the approximation to G_l has greater error toward the center of the graph; (2) the error is exaggerated by the fact that the distance from the "bottom" of the y-axis to the origin is of the same order as the center errors. In experiment 2 the distance is .929 and $\mu(19) - \Lambda_0^{2000}(19, 0) = .02963$ (see consideration 1 and 2 of Section 7).

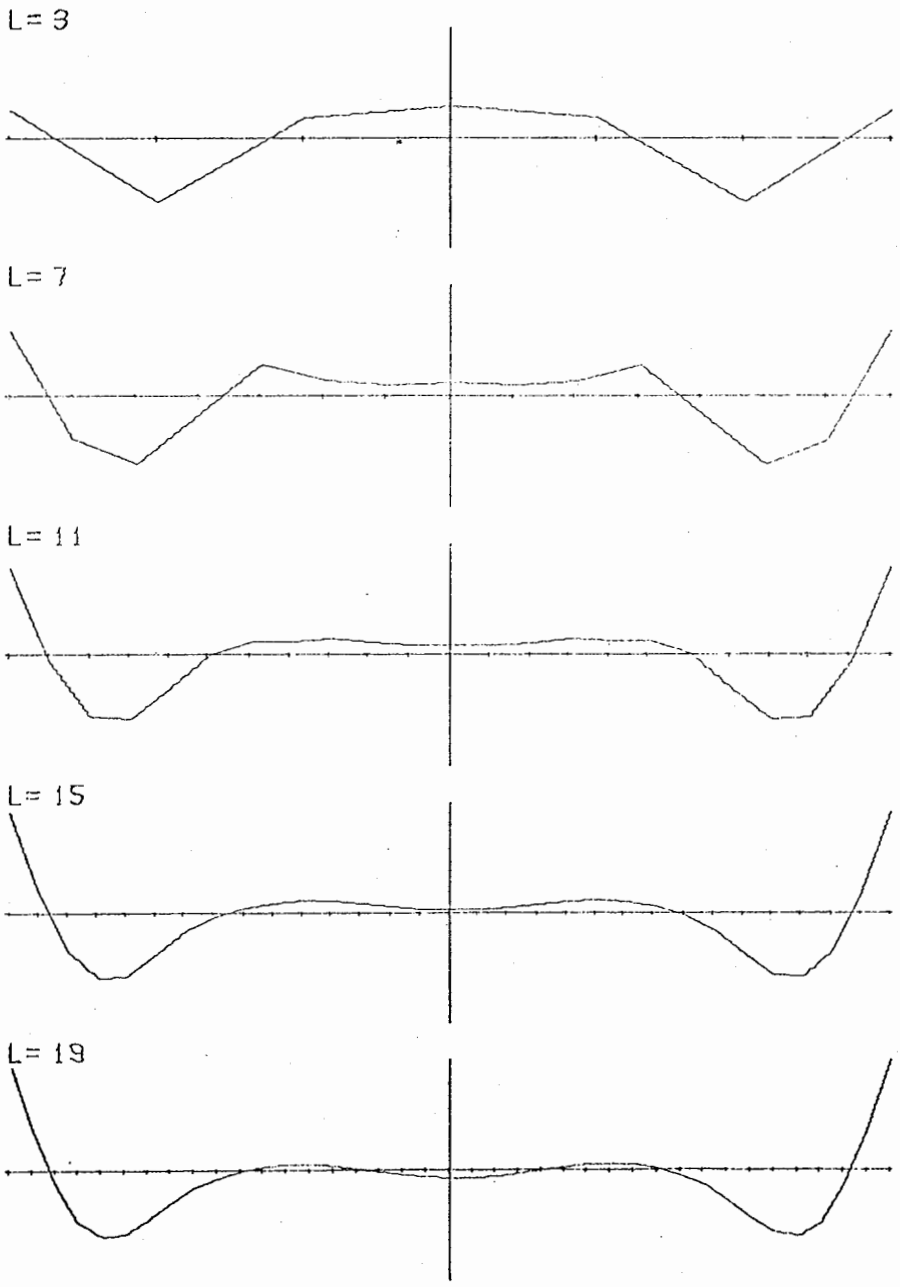
Experiment 2



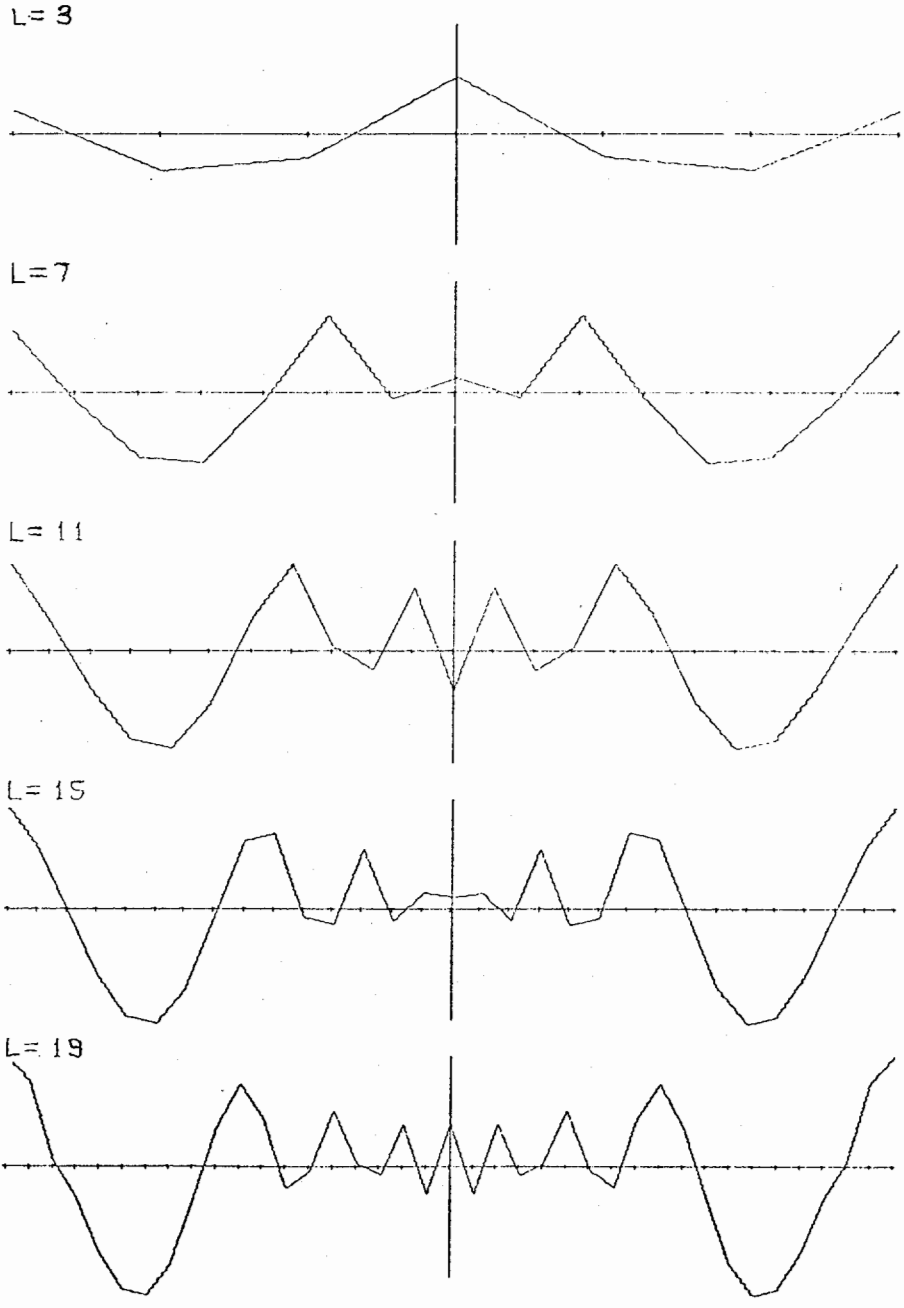
Experiment 3



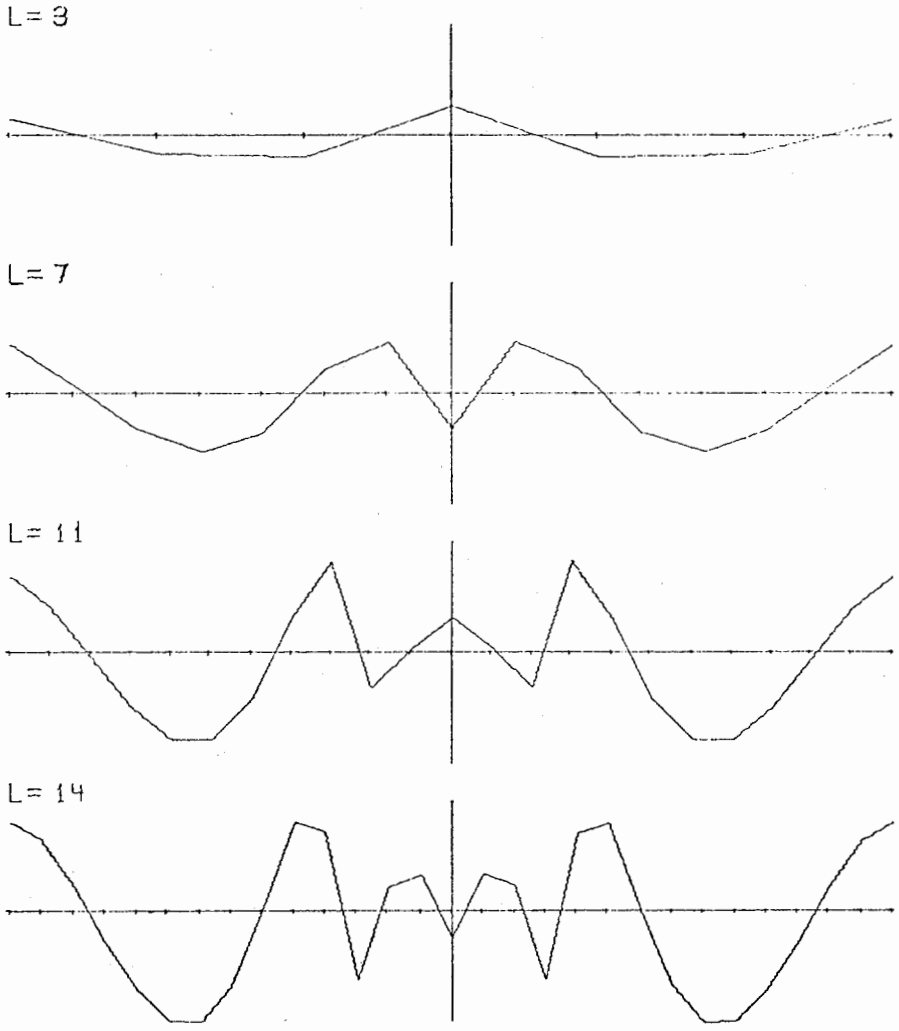
Experiment 4



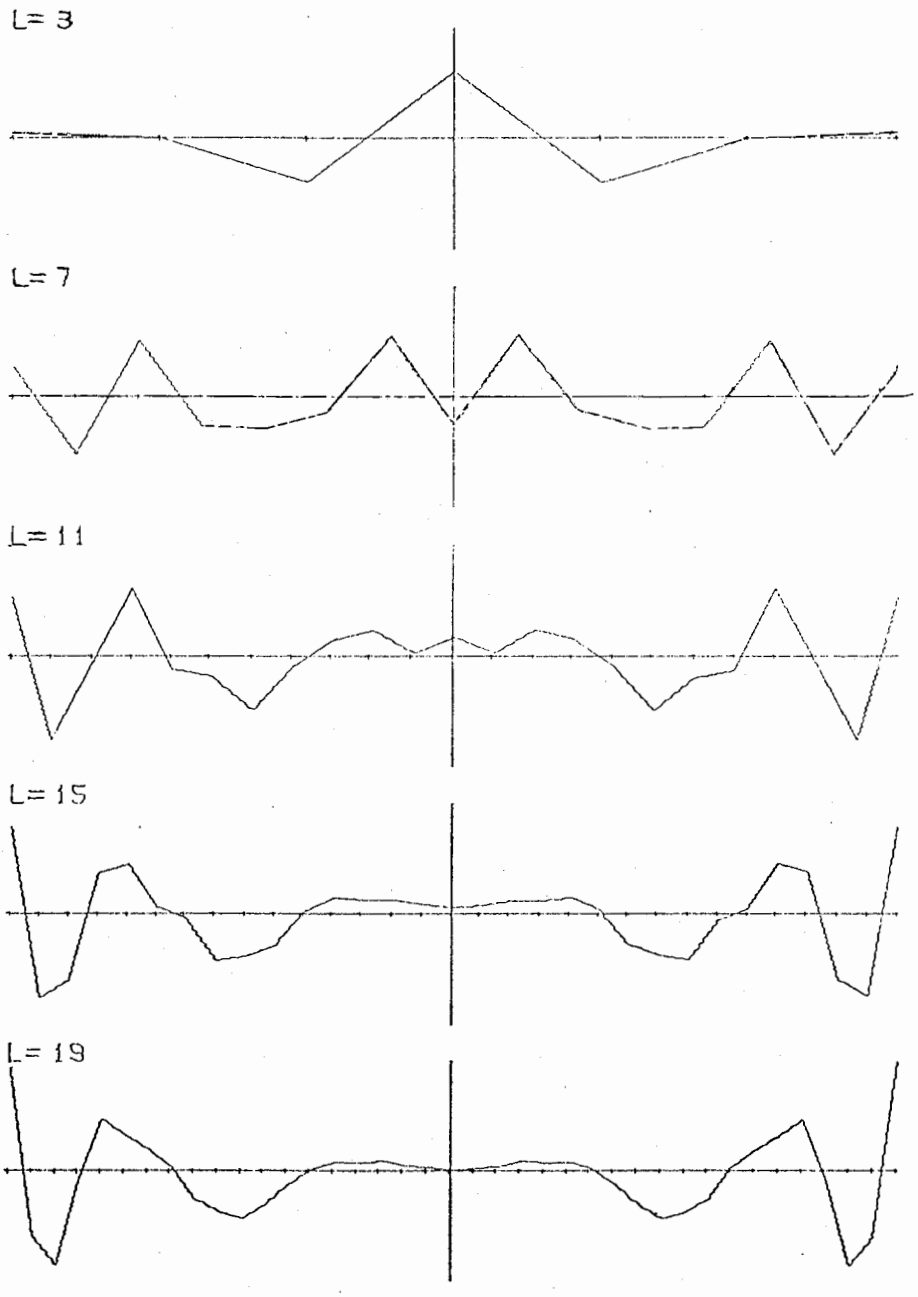
Experiment 5



Experiment 7



Experiment 9



10. Results for non-Zoll experiments

Two non-Zoll experiments were made, 8 and 10. In both experiments the perturbing functions are even, that is, $\sigma(x) = \sigma(-x)$ for $-1 \leqslant x \leqslant 1$. Recall that $\sigma(x)$ must be odd for the perturbation of Δ to the Zoll. $x^2 - 1$ is the perturbing function for 10, and $1 - x^2$ for 10.

If $\sigma(x)$ is odd, then for equal values of ε the metrics induced by $\sigma(x)$ and $-\sigma(x)$ are isometric. Thus the spectrum of the corresponding Laplacians are the same. If $\sigma(x)$ is not odd, then the metric for $\sigma(x)$ and $-\sigma(x)$ are not necessarily isometric. This fact is mentioned to indicate that the Laplacians for the two non-Zoll experiments are essentially different.

The initial cluster for both 8 and 10 has only one member, .0000. This is the same situation for Zoll experiments.

In experiment 8 for each $l > 0$ the l-th cluster displays the following pattern:

$$\Lambda_\varepsilon^n(l, 0) < \Lambda_\varepsilon^n(l - 1, 1) < \cdots < \Lambda_\varepsilon^n(0, l) < \mu(l).$$

This experiment also shows that $\Lambda_\varepsilon^n(l, 0) - \mu(l, 0) - \mu(l)$ gets larger as l gets larger. $\Lambda_\varepsilon^n(l, 0) - \mu(l)$ gets large so fast that the clusters eventually overlap each other, that is, there is an l_0 such that for $l > l_0$, $\Lambda_\varepsilon^n(l + 1, 0) < \Lambda_\varepsilon^n(0, l)$. In fact, the minimum member of the 8-th cluster, $\Lambda_\varepsilon^n(8, 0) = 34.11927$, is less than the maximum member of the 6-cluster, $\Lambda_\varepsilon^n(0, 6) = 39.21949$.

In experiment 10 for each $l > 0$ the l-th cluster displays the following pattern:

$$\Lambda_\varepsilon^n(l, 0) > \Lambda_\varepsilon^n(l - 1, 1) > \cdots > \Lambda_\varepsilon^n(0, l) > \mu(l).$$

This pattern is the reverse of the 8 pattern. Again the clusters eventually overlap. The maximum member of the 16-th cluster, $\Lambda_g^n y(16, 0) = 425.12273$, is greater than the minimum of the 19-th cluster, $\Lambda_\varepsilon^n(0, 19) = 391.90565$.

If the patterns of these two non-Zoll experiments are persistent, then the following statements would be true:

(1) the diameter of the l-th cluster union the set containing $\mu(l)$ is not bounded:

(2) Conjecture 1 does not hold.

APPENDIX; LISTING OF SUBROUTINE EIGEN2

What follows is a listing of SUBROUTINE EIGEN2. This subroutine is the part of the computer program which actually does computation described in §4. The language of the subroutine is FORTRAN.

Basically this subroutine first computes the nonzero elements of the tridiagonal matrix which is to be diagonalize. DIAG(K) = β_K are the diagonal elements ($K = 1, \cdots, N-1$ for $m \neq 0$, and $K = 1, \cdots, N+1$ for $M = 0$). OFF(K) is the product of the corresponding elements of the super- and sub-diagonals ($K = 2, \cdots, N-1$ for $M \neq 0$, and $K = 2, \cdots, n+1$ for $M = 0$). OFF(K) = $\alpha_K \gamma_{K-1}$.

Next, SUBROUTINE EQRET1S is called. This subroutine calculates the first IFIRST eigenvalues of the tridiagonal matrix, and stores the values in DIAG(K), $K = 1, \cdots,$ IFIRST.

In this particular version SIG(X) = $1 - X^2$. To make experiments with different perturbing functions SIG(X) and its derivative, DSIG(X) must be defined by the appropriate functions.

```
      SUBROUTINE EIGEN2
C
C     INPUT      N = NUMBER OF EQUAL SUBDIVISIONS
C                M = DIFF. EQ. PARAMETER = 0, 1, 2, 3, · · ·
C                EPSILON = DIFF. EQ. PARAMETER NEAR ZERO
C                IFIRST = NUMBER OF EIGENVALUES TO BE COM-
                 PUTED
C     OUTPUT     DIAG(K) = K-TH EIGENVALUE (K = 1, IFIRST)
      COMMON     N, M, EPSILON, DIAG(5000), IFIRST
      DIMENSION OFF (5000)
      SIG(X) = 1 - X*X
      DSIGN(X) = -2*X
C
C     DEFINE DIAG, OFF
C
C
      PI = 3,14159 26535 89793
      GAMMA = 0
      N1 = N - 1
      DELTA = PI/N
      DO 10 K = 1, N1
      THETA = K*DELTA
      STHETA = SIN(THETA)
      CTHETA = COS(THETA)
      H = 1 - EPSILON*SIG(CTHETA)
      DH = EPSILON*DSIG(CTHETA)*STHETA
```

```
      A = M**2/STHETA**2
      B = - CTHETA?(STHETA*H**2) + DH/H**3
      C = -1/H**2
      ALPHA = -B/(2*DELTA) + C/DELTA**2
      IF(K.EQ.1)ALPHA1 = ALPHA
      BETA = A - 2*C/DELTA**2
      DIAG(K) = BETA
      OFF(K) = ALPHA*GAMMA
      IF(OFF(K).LT.0) GO TO 20
      GAMMA = B/(2*DELTA) + C/DELTA**2
      IF(K, EQ.N1) GAMMAN1 = GAMMA
C
C     NOTICE GAMMA IS DEFINED AFTER GAMMA IS USED TO DEFINE
C     OFF(K)
C     THE REASON, GAMMA AT THE (K - 1)-STEP IS USED TO DEFINED
C     OFF(K)
C
 10   CONTINUE
      IF (M.GT.0) TO TO 30
C
C     SETUP FOR M = 0
C
      X = 4/DELTA**2
      BFTAO = X
      GAMMA0 = -X
      ALPHAN = -X
      BETAN = X
C
C     REDEFINE DIAG
C
      DIAG(N + 1) = BETAN
      DO 40 K = 1, N1
      L = N - K
C     L = N - 1, N - 2, · · · , 2.1
      DIAG(L + 1) = DIAG(L)
 40   CONTINUE
      DIAG(1) = BETAO
C
C     REDEFINE OFF
C
```

```
      OFF(N + 1) = ALPHAN*GAMMAN1
      DO 50 K = 2, N1
      L = N - K + 1
C     L = N - 1, N - 2, · · · , 3.2
      OFF(L + 1) = OFF(L)
 50   CONTINUE
      OFF(2)-ALPHA1*GAMMA0
C
      N1 = N + 1
 30   CONTINUE
C
C     SETUP TO USE LIBRARY    SUBROUTINE EQRT1S
C       EQRT1S PARAMETERS       EIGEN2 PARAMETERS
C       D                       DIAG
C       E2                      OFF
C       N                       N1
C       M                       IFIRST(.LT.N1)
C       ISW                     1
C       IER                     IER
      CALL EQRT1S(DIAG.OFF,N1,IFIRST,1,IER)
      RETURN
 20   CONTINUE
      PRINT 910, N, M, EPSILON
      PRINT 920, K, OFF(K), A, B, C, ALPHA, BETA, GAMMA
      STOP
 910  FORMAT(1H-,T11,*PARAMETERS*,10X,2HN = ,I4,10X,2HM = ,I2,10X,
     *              *EPSILON = *, F9.5//)
 920  FORMAT(T11,*PROGRAM STOPPED BECAUSE OF ERROR
     *              OFF(K).LT.ZERO*//
     *              T11,*OFF(*,I3,2H) = ,E16.10,5X,2HA = ,E16.10,5X,2HB
     *              = E16.10,5X,2HC = ,E16.10//T11,*ALPHA = *,E16.10,
     *              5X,*BETA*,E16.10,5X,*GAMMA = *,E16.10)
      END
```

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